Short Papers

Analysis of an Oval Symmetrically Located Inside a Rectangular Boundary by Conformal Mapping

B. N. DAS, K. V. SESHAGIRI RAO, AND A. K. MALLICK

***Abstract*—This paper describes the analysis of a transmission line having an oval-shaped center conductor symmetrically placed inside an outer conductor in the form of a rectangular waveguide. A conformal transformation is used to calculate the characteristic impedances of oval, elliptic, circular, and planar conductors. The impedance data of these structures are presented in the form of charts for different aspect ratios of the rectangular outer conductor. The charge distribution on the center conductor is also determined.**

I. INTRODUCTION

A few investigations have been carried out on the analysis of transmission lines having a circular conductor located inside a metallic square, inside a trough, and between infinite parallel planes [1]–[3]. Tippet found the impedance of a transmission line consisting of a septum placed inside a structure in the form of a rectangular waveguide [4]. The analysis of a transmission line having a conductor of elliptic cross section asymmetrically located between infinite parallel plates has also been reported recently [5].

In the present work, analysis based on quasi-static approximation is presented for the case of a transmission line having an oval-shaped center conductor located symmetrically inside a conductor with rectangular cross section. The generalized conformal transformation is therefore limited to the fundamental TEM mode and can be used to determine the characteristic impedance of a transmission line having a center conductor in the form of 1) an ellipse with the principal axes parallel to either boundary of the rectangular outer conductor, 2) a circle, and 3) a symmetrically oriented septum.

The parametric equations which describe the oval-shaped boundary of the center conductor are obtained from a conformal transformation. The data on the characteristic impedance are presented in the form of charts, with varying parameters from which the impedances of all the above structures can be determined. The charge distribution on the center conductor for various eccentricities is also determined.

II. ANALYSIS

Consider a center conductor with a curved boundary placed symmetrically inside a conductor of rectangular cross section as shown in Fig. 1(a). Because the structure is symmetric about the x axis, the analysis is carried out only for the upper hatched portion of the structure for which the conformal transformation can be found. The method of obtaining the conformal transformation of such a polygon with curved boundaries has been suggested by including an appropriate curve factor in the equation representing the conformal transformation [6]. The hatched portion of Fig. 1(a) can therefore be treated as a curvilinear polygon with vertices at the points A, B, C, E, F, and G of Fig. 1(a). Using the combination of both Schwarz–Christoffel and Joukowski transformations, the shaded region of Fig. 1(a) is transformed into the upper half plane of Fig. 1(b) (t plane), leads to an equation of the form [6]

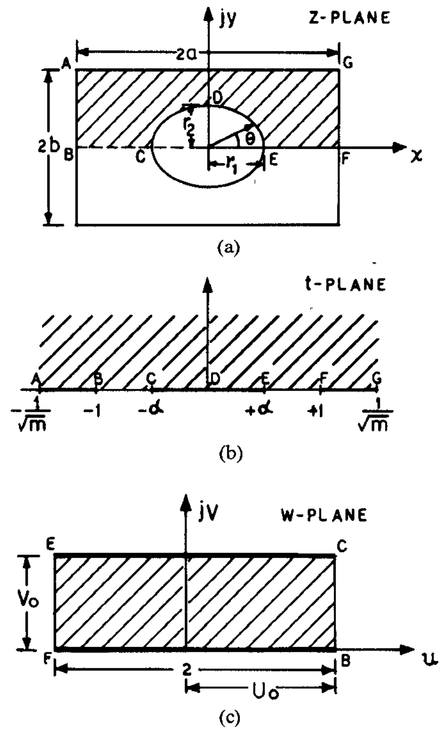

Fig. 1. Transmission-line configuration and its conformal representation.

$$\frac{dZ}{dt} = C_1 \cdot \frac{t + \lambda\sqrt{t^2 - \alpha^2}}{\sqrt{(t^2-\alpha^2)(1-t^2)(1-mt^2)}}. \tag{1}$$

Carrying out an integration in terms of elliptic integrals, (1) takes the form

$$Z = x + jy = C_1\left[\lambda F(\sin^{-1} t|m) - \frac{j}{\sqrt{1-m\alpha^2}} \cdot F\left(\sin^{-1}\sqrt{\frac{1-\alpha^2}{1-mt^2}}\,\middle|\, g\right)\right] + C_2 \tag{2}$$

where C_1, C_2, λ, m, and α are constants

$$g = \frac{(1-m)}{(1-m\alpha^2)}$$

and F is the incomplete elliptic integral of the first kind of a given argument and modulus.

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B. N. Das and A. K. Mallick are with the Department of Electronics and Electrical Communication Engineering, Indian Institute of Technology, Kharagpur-721302, India.

K. V. S. Rao is with the Centre for Research and Training in Radar and Communication, Indian Institute of Technology, Kharagpur-721302, India.

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Substituting the coordinates of the points A, B, C, D, E, F, and G in both planes (Z and t) in (2) and solving the resulting set of equations, the constants C_1, C_2, r_1, r_2, λ, α, m, a, and b are found to be related as follows:

$$C_1 = \frac{a\sqrt{1-m\alpha^2}}{\left[K'(g)+\sqrt{1-m\alpha^2}\lambda K(m)\right]} \tag{3a}$$

$$C_2 = \frac{jaK(g)}{\left[K'(g)+\sqrt{1-m\alpha^2}\lambda K(m)\right]} \tag{3b}$$

where K is the complete elliptic integral of the first kind at modulus m or g, K' is the associated complete elliptic integral corresponding to the complementary modulus

$$\frac{b}{a} = \frac{\left[K(g)+\lambda\sqrt{1-m\alpha^2}K'(m)\right]}{\left[K'(g)+\lambda\sqrt{1-m\alpha^2}K(m)\right]} \tag{3c}$$

$$\frac{r_1}{a} = \frac{\lambda\sqrt{1-m\alpha^2}\cdot F(\sin^{-1}\alpha|m)}{\left[K'(g)+\lambda\sqrt{1-m\alpha^2}K(m)\right]} \tag{3d}$$

$$\frac{r_2}{a} = \frac{\left[K(g)-F\left(\sin^{-1}\sqrt{1-m\alpha^2}|g\right)\right]}{\left[K'(g)+\lambda\sqrt{1-m\alpha^2}K(m)\right]} \tag{3e}$$

$$\lambda = k\cdot\frac{\left[K(g)-F\left(\sin^{-1}\sqrt{1-m\alpha^2}|g\right)\right]}{\sqrt{1-m\alpha^2}\cdot F(\sin^{-1}\alpha|m)} \tag{3f}$$

where k (the compression ratio) $= r_1/r_2$.

The curved boundary CDE of Fig. 1(a) (Z plane) is transformed into a planar conductor CDE of Fig. 1(b) (t plane). For any point on the curve CDE, t is real and its magnitude is less than α.

The boundary of the conductor is assumed to be in the form of a generalized oval which is a closed curve symmetric about a pair of axes and concave toward the center bounding a convex domain [7].

This curve satisfies the parametric equations of the form

$$x = \frac{r_1 F(\sin^{-1}t|m)}{F(\sin^{-1}\alpha|m)} \tag{4a}$$

$$y = \frac{r_2\left[K(g)-F\left(\sin^{-1}\sqrt{\frac{1-m\alpha^2}{1-mt^2}}\,\middle|\,g\right)\right]}{\left[K(g)-F\left(\sin^{-1}\sqrt{1-m\alpha^2}|g\right)\right]}. \tag{4b}$$

From the computed results, it is found that for the aspect ratio (b/a) different from unity, (4a) and (4b) represent an ellipse for the values of α^2 less than 0.99 and they represent oval for the values of α^2 greater than 0.99. For a value of aspect ratio equal to unity, the center conductor retains the shape of the ellipse even for the values of α^2 up to 0.999.

Depending upon the orientation of the principal axes, the eccentricities of the generalized ellipse are given by

$$e_1 = \sqrt{1-\left(\frac{r_2}{r_1}\right)^2}, \qquad r_1 > r_2 \tag{5a}$$

$$e_2 = \sqrt{1-\left(\frac{r_1}{r_2}\right)^2}, \qquad r_1 < r_2. \tag{5b}$$

The upper half of Fig. 1(b) is transformed into the parallel plate configuration of Fig. 1(c) (W plane) using the transformation [8]

$$W = u + jv = C_3\int_0^{t/\alpha}\frac{dt'}{\sqrt{(1-t'^2)(\alpha^2-t'^2)}} + C_4$$

$$= C_3 F(\Phi|\alpha^2) + C_4 \tag{6}$$

where $\Phi = \sin^{-1}t/\alpha$, and C_3 and C_4 are constants.

From a substitution of the coordinates of the points B, C, E, and F in the t and W planes, the transformation takes the form

$$W = u + jv = -\frac{F(\Phi|\alpha^2)}{K(\alpha^2)} + j\frac{K'(\alpha^2)}{K(\alpha^2)} \tag{7}$$

with U_0 and V_0 shown in Fig. 1(c) given by

$$U_0 = 1 \tag{8a}$$

$$V_0 = \frac{K'(\alpha^2)}{K(\alpha^2)}. \tag{8b}$$

III. Evaluation of the Characteristic Impedance

One half of the structure shown in Fig. 1(a) has been transformed into the parallel plate configuration shown in Fig. 1(c). The width of the parallel plates is 2, and V_0 is the separation between them. The total capacitance per unit length of the line is twice that of the parallel plate capacitor of Fig. 1(c) and is

$$C' = \frac{4\epsilon_0\epsilon_r}{V_0}. \tag{9a}$$

The characteristic impedance of the transmission line is

$$Z_0 = \frac{30\pi}{\sqrt{\epsilon_r}}\cdot\frac{K'(\alpha^2)}{K(\alpha^2)}. \tag{9b}$$

The dependence of the characteristic impedance on the dimensions of the transmission line can be determined from (3a)–(3f) and (9b). For a given impedance of the line, the modulus α of the elliptic integrals can be obtained by solving the transcendental equation (9b). From a knowledge of α, the compression ratio k and the aspect ratio, the value of m can be obtained from (3f) and (3c). Knowing m, r_1/a and r_2/a can then be obtained from (3d), (3e), and (3f). The remaining constants C_1 and C_2 follow from (3a) and (3b), respectively. The shape of the conductor depends upon the compression ratio k which is a function of m and α. It is found from (3f) that the parameter λ depends upon m and α. Thus, the impedance and the conductor shape depends upon m and α.

Results are presented in Figs. 2 and 3 as constant impedance contours for the values of b/a equal to 1 and 2, respectively. From a knowledge of the compression ratio k, e_1, or e_2 can be found from (5a) and (5b), respectively. $e_1 =$ const, $e_2 =$ const, or $k =$ const are straight lines passing through the origin of the ($r_1/a - r_2/a$) plane. The intersection of the straight line for which $e_1 = e_2 = 0$ or $k = 1$ with the constant impedance contours gives the impedance of 1) conductors having a circular cross section and 2) the impedances of ovals with identical principal axes located symmetrically inside a rectangular boundary.

The intersection of straight lines making angles other than 45° with the r_1/a or r_2/a axis and constant impedance contours covers the cases of symmetrically located elliptic conductors and oval-shaped conductors with unequal principal axes.

A. Elliptic Conductor

The impedance data for the case of the elliptic conductor with various eccentricities are presented in Figs. 2 and 3.

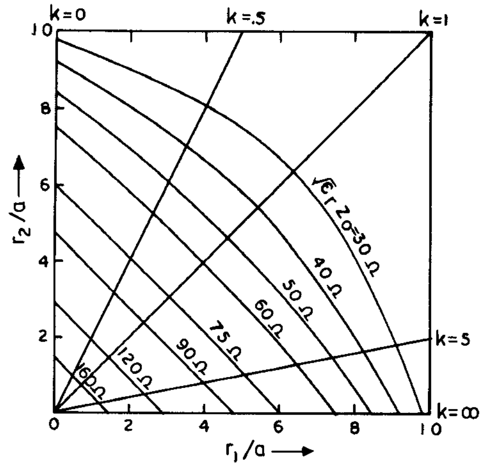

Fig. 2. Variation of impedance of a transmission line as a function of conductor dimensions for $b/a = 1$.

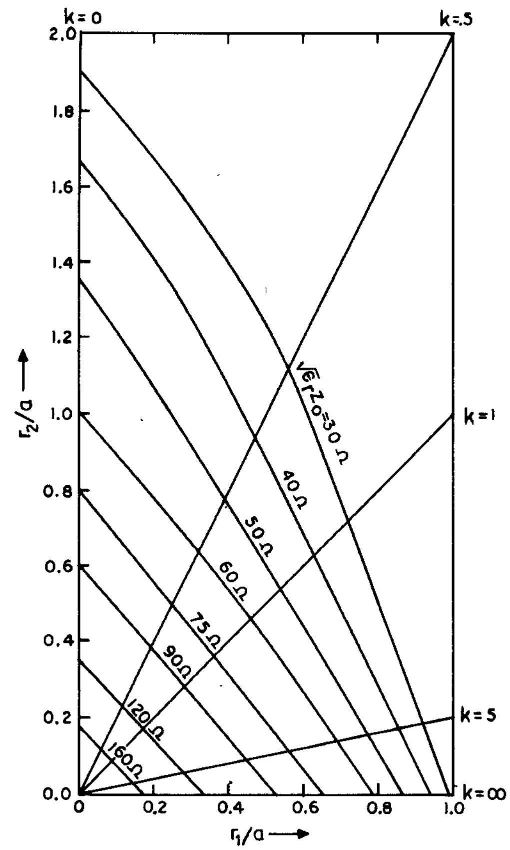

Fig. 3. Variation of impedance of a transmission line as a function of conductor dimensions for $b/a = 2$.

B. Septum Inside a Rectangular Boundary

When the compression ratio is either 0 or ∞, the oval degenerates into a straight line parallel to either side of the rectangular boundary.

For $k = 0$, it is found from (3d)–(3f) and (5b) that

$$\frac{r_1}{a} = 0 \quad e_2 = 1 \quad \lambda = 0$$

and

$$\frac{r_2}{a} = \frac{\left[K(g) - F\left(\sin^{-1}\sqrt{1-m\alpha^2}\,|g\right)\right]}{K'(g)}. \qquad (10)$$

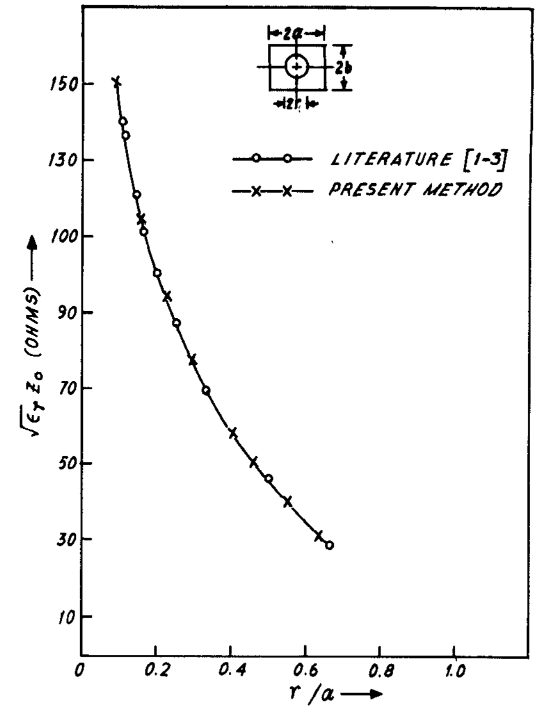

Fig. 4. Variation of impedance of a transmission line having a circular inner conductor inside a square outer conductor as a function of radius ($r_1 = r_2 = r$) for $b/a = 1$.

The corresponding transformation is given by

$$Z = \frac{ja\left[K(g) - F\left(\sin^{-1}\sqrt{\dfrac{1-m\alpha^2}{1-mt^2}}\,\middle|\,g\right)\right]}{K'(g)}. \qquad (11)$$

For $k \to \infty$, it is found from (3d)–(3f) and (5a) that

$$\frac{r_2}{a} = 0 \quad e_1 = 1 \quad \lambda \to \infty$$

and

$$\frac{r_1}{a} = \frac{F(\sin^{-1}\alpha|m)}{K(m)}. \qquad (12)$$

The corresponding transformation for this case is found to be

$$Z = \frac{a}{K(m)} F(\sin^{-1} t|m). \qquad (13)$$

Both (11) and (13) are of the same form as obtained by Tippet [4].

C. Circular Conductor

$e_1 = e_2 = 0$ or $k = 1$ corresponds to the case of a circular conductor symmetrically located inside a rectangular boundary. By substituting $k = 1$ in (3f) the value of λ is found to be

$$\lambda = \frac{\left[K(g) - F\left(\sin^{-1}\sqrt{1-m\alpha^2}\,|g\right)\right]}{\sqrt{1-m\alpha^2}\,F(\sin^{-1}\alpha|m)}.$$

By using this value of λ in (3a) through (3e) and (2), the conformal transformation for the case of a circular conductor is obtained.

The comparison of impedance data for a circular inner conductor inside a square outer conductor obtained by the present method with those published in the literature [1]–[3] is presented in Fig. 4, and also an additional data for the case of a circular

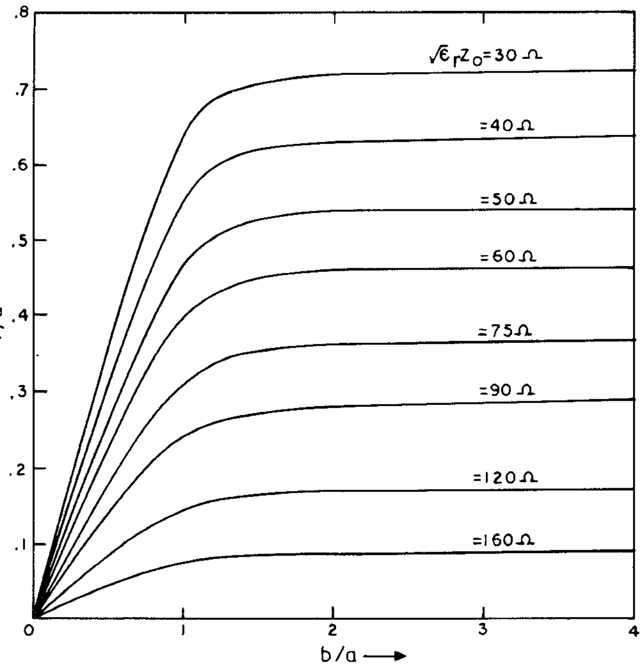

Fig. 5. Dependence of the radius ($r_1 = r_2 = r$) of the transmission line with inner circular conductor as a function of aspect ratio of the outer conductor for various values of the impedances.

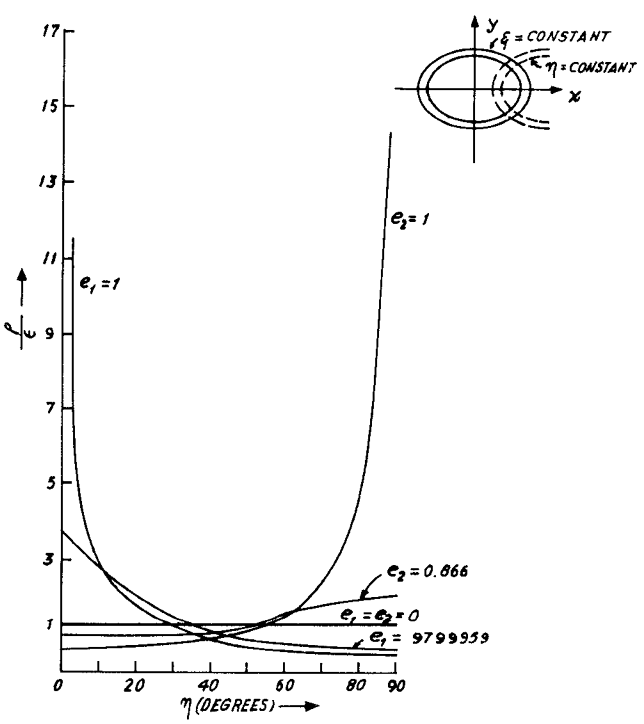

Fig. 6. Charge distribution on the center conductor with e_1 or e_2 as parameter for $b/a = 1$ and $Z_0 = 51.56\ \Omega$

inner conductor showing the relation between the radius of the inner conductor and the aspect ratio of the outer conductor for different values of the characteristic impedance is presented in Fig. 5.

IV. Charge Distribution on the Center Conductor of the Transmission Line

The charge density on the surface of the center conductor is given by

$$\rho_s = \epsilon E_n \tag{14}$$

where E_n is the normal component of the electric field on the boundary of the center conductor.

The electric field in the cross section of the transmission line is given by [9]

$$E = -\left(\frac{dW}{dZ}\right)^* \tag{15}$$

where the asterisk denotes a complex conjugate.

Using (2), (7), (14), and (15), the charge distribution on the center conductor is evaluated for $b/a = 1$, $\sqrt{\epsilon_r}\,Z_0 = 51.56\ \Omega$, and different values of e_1 and e_2. The charge distribution so determined is presented in Fig. 6.

V. Conclusions

It has been possible to determine the impedance and charge distribution for conductors of different shapes placed inside a conductor with rectangular outer boundary using a common general formulation. The curves of Fig. 4 reveal good agreement between the results for a rectangular coaxial line with circular inner conductor obtained by the present method and those obtained by the other methods [1]–[3]. For a specified impedance level, the shape of the conductor and the corresponding parameters (m, λ, α^2) can be determined with good accuracy.

The results of the analysis can be used to determine the dimensions of the center conductor for a desired characteristic impedance and specified aspect ratio (b/a). For a particular value of impedance, the form of the center conductor can be an ellipse, circle, or a septum. The advantage of coaxial lines with elliptic or circular inner and rectangular outer conductors over other forms of the lines is that they have a higher breakdown voltage. The power handling capacity of these lines is therefore higher.

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